Analytical Technique for Non-Uniformly Prestressed Tapered Beams with Exponentially Varying Thickness Resting on Vlasov Foundation under Variable Harmonic Load

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Abstract: *In this paper, the motion of non-uniformly prestressed tapered beams with exponentially varying thickness resting on Vlasov foundation under variable harmonic load moving with constant velocity is investigated. The governing equation is a fourth order partial differential equation. The solution technique is based on the method of Galerkin with series representation of Heaviside function, Struble's asymptotic method and Laplace transformation technique in conjunction with convolution theory. The result shows that, an increase in the values of the structural parameters such as foundation stiffnesses, axial force, moment of inertia of the beam and exponential factor reduces the response amplitude of the beam for the dynamic problem. Furthermore, it is found that the moving force solution is not always an upper bound for the accurate solution for the non-uniformly prestressed tapered beams*.

Keywords: non-uniformly prestressed tapered beams, exponentially varying thickness, Vlasov foundation, variable harmonic load.

# **Introduction**

Studies in structural dynamics dealing with moving loads on elastic structures are enormous and have been enriched in the last few decades by the development of high-speed railway networks, elevated roadways, highway bridges especially cable-stayed and suspension bridges etc., in the developed and developing countries. The fundamental problems have been the increasing high speeds and weights of the vehicles, which this structure carries. In particular, Krylov [1] first studied the dynamic response of a simply supported beam, traversed by a constant force moving at a uniform speed. His results were obtained using the method of Eigen-functions. He assumed that the mass of the load is smaller than that of the beam. Later, Timoshenko [2] used energy methods to obtained solution in series form for simply supported finite beam on an elastic foundation subjected to time dependent point loads moving with uniform velocities across the beam. Kenny [3] similarly investigated the dynamic response of infinite beams on an elastic foundation under the influence of loads moving at constant speeds. He included the effects of viscous damping in the governing differential equation of motion. Steel [4] also investigated the response of a finite simply supported Bernoulli-Euler beam to a unit force moving at a uniform velocity. He analyzed the effects of this moving force on beams with and without an elastic foundation using a considerable simpler vector formulation with a Laplace rather than Fourier transformation. In a much later development Oni, [5] considered the problem of a harmonic time-variable concentrated force moving at a uniform velocity over a finite deep beam. The methods of integral transformations were used. Series solution, which converges, was obtained for the deflection of simply supported beams and analyzed for various speeds of load. Just as for elastic beams, many authors have tackled the problem of dynamic response of an elastic plate to moving loads when the mass effect of the moving load is neglected. The study of non-uniformly prestressed tapered beams with exponentially varying thickness resting on Vlasov foundation under variable harmonic load moving with constant velocity forms a very important structural element in engineering design and construction. It has also become the objective of various researchers in the field of Applied Mathematics. In general, problems of this type are mathematically complex if analytical approach is used. Thus, most of the research works available in the Literature are those in which Numerical technique is used. This is due to great amount of computational labour, which is required to both set up and solve the necessary equations. A major break-through in this field of research is the work of Timoshenko [2] who gave impetus to research work in this area of study. The analysis of the dynamic response of a simple beam continuously supported by a viscoelastic foundation to a moving load, moving at variable speed was considered. The analysis reveals several resonance conditions depending on the viscoelasticity of the foundation. In addition, a theory for the response to an arbitrary number of concentrated moving masses of a rectangular plate continuously supported by an elastic Pasternak type foundation was developed [6]. It was found that the critical speeds of the system increased with increase in the values of the foundation moduli whether the inertia of the moving load is considered or not. The displacement response of a simply supported non-uniform beam resting on an elastic foundation to several moving load was later taken up, and concluded that the maximum transverse deflection of the beam is always greater than the displacement of the moving mass [7]. A modification of the asymptotic method was used to simplify the resulting sequence of differential equation [8]. Sayad et al. Study vibration analyses of a tapered beam with exponentially varying thickness resting on Winkler foundation using the differential transform method [9]. It is well known that in the dynamical system like this, analytical are desirable as a method of solution, as there often shed light on vital information about the vibrating system.

Thus, this paper studied the dynamic response of non-uniformly prestressed tapered beams with exponentially varying thickness resting on Vlasov foundation under variable harmonic loadusing analytical approach. Several numerical examples will also be presented. It is assumed that the speed at which the load traverses the structural element is constant.

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# **Mathematical Model**

## **Vibration equation**

The problem of moving harmonic time-variable load on simply supported tapered beams with exponentially varying thickness of finite length L resting on Vlasov foundation shown in fig. 1 is given as

Where is the distance along the beam; *t* is the time; *I(x)* is the moment of inertia of the beam cross section at a distance *x;* ρ is the mass density per unit volume; *E* is beam modulus elasticity; *A(x)* is the cross sectional area of the beam at a distance *x*; *V(x,t)* is the beam lateral displacement; *N(x)* is the external axial load acting on the beam cross section at a distance *x*; *K1* and *K2* are the foundation stiffnesses per unit length of the beam in the Vlasov model soil.



**Fig. 1:** *Geometry of a tapered beam with exponentially varying thickness on Vlasov foundation*.

For a constant *E* and a variable cross section with respect to the horizontal axes *(x)*, as shown in fig.1 we can expand Eq. (1) to obtains

Since the breadth and depth of the beam i.e. *b* and *a* are , respectively. One can obtains

Where φ is a factor to show the exponential rate of the beam. Substituting Eq. (3) into (2), yields

Furthermore, we adopt example of the work of Kien [10] to define the prestressed parameter in (4) with constant axial force *No* at the left end of the beam as

However, it is assumed that the load moves at a constant velocity *(c)*, keep contact with the beam continually before arriving at the other end support and it is harmonic time-variable load. Hence, taking also cognizance both the gravitational and inertia load, externally applied moving load *f(x, t)* is defined as

Where *M* is the mass of the moving load, *ω* is circular frequency of the harmonic load and *g* is the acceleration due to gravity.

For simply supported tapered beam of finite length *L*, the boundary conditions may be described mathematically as

The tapered beam is considered to be initially at rest. Hence the corresponding initial boundary conditions is

In view of Eqs. (5) and (6), after some simplification and rearrangement, Eq. (4) becomes

# **Method of Solution**

## **Galerkin's method**

To solve the dynamic problem (9), we shall use the versatile solution technique called **Galerkin's method**, which often uses to solve diverse problems in dynamic of structure. Thus, we use the Galerkin's method described in Oni and Awodola [11] to reduce the fourth order partial differential equation to a sequence of fourth order ordinary differential equation.

Thus a solution of the form

Is sought, is chosen as a suitable kernel of the Galerkin's method in (9) such that the boundary conditions given are satisfied. Therefore substituting eq. (10) into (9), one obtains

In order to determine an expression for *Zm (t)*, it is required that the expression on the left hand side of (11) is orthogonal to the function *Uk (x)*. Thus, we have

Where

In view of boundary condition, is chosen as

Furthermore, using the property of the Heaviside function, it can be expressed in series form given by [3]

Therefore using eq. (15),

Becomes

Where

Substituting eqs. (14), (16) and (17) into the coefficients of eq. (12), after evaluating the integrals in (13) and rearranging eq. (12), one obtains

Where

Equation (18) is now the fundamental equation governing the dynamic problem of non-uniformly prestressed tapered beam with exponentially varying thickness resting on Vlasov foundation under variable harmonic load. It follows that two special cases of (18) arise, namely '**moving force**' and '**moving mass**' problems.

## **Non-uniformly Prestressed Tapered Beam Traversed by Moving Force**

In this section, an approximate model of the differential equation describing the response of the elastic structure is obtained by neglecting the inertial terms i.e. .

Thus, eq. (18) is of the form

Where

Further simplification of (19) yields

Where

Subjecting the system of ordinary differential equation (20) to a Laplace transform defined as

In conjunction with the initial conditions defined in (8), gives the following simple algebraic equation

Thus, to obtain Laplace inversion of (22), we adopt the following representations

So that the Laplace inversion of is the convolution of (22) defined as

Thus, the Laplace inversion of (22) is given as

Where

In view of eq. (10) after evaluating the integrals in (23) one obtains

Equation (24) represent the transverse displacement of the non-uniformly prestressed tapered beam with exponentially varying thickness resting on Vlasov foundation under variable harmonic moving force.

## **Non-uniformly Prestressed Tapered Beam Traversed by Moving Mass**

In this section, the solution to the entire equation (18) is sought when no terms of the coupled differential equation is neglected. Therefore, the equation is rearranged to take the form

In order to further simplify (25), we apply cosine series expansion defined as

For small value of ω, eq. (25.1) becomes

Therefore, eq. (25) becomes

Evidently, unlike the moving force problem, an exact analytical solution to (25) is not possible. Though the equation yields readily to numerical technique, an analytical approximation method is desirable as a solution so obtained often-shed light on vital information about the vibrating system. Thus, we resort to a modification of the **asymptotic** method due to **Strubble**, which is often used in treating oscillatory system. To this ends, using this method, one seek the modified frequency of the free system due to the presence of the effect of moving mass. An equivalent system operator defined by the modified frequency then replaces (25). Thus, a parameter *∈<1* is considered for any arbitrary mass ratio defined by

Clearly, all the various time independent coefficients of the differential operator, which acts on in (25) can be written in terms of ∈ i.e.

Whenever

Substituting eqs. (28) and (29) into (25), after some rearrangement yields

Setting *∈ = 0*, eq. (30) resort to a solution corresponding to the case in which the inertia effect of the mass system is regarded as negligible, then the solution of (30) becomes

Where are constants. Since *∈<1*, Strubble's technique requires that the asymptotic solution to the homogeneous part of (30) be of the form

Where are slowly varying function of time.

The variational equations describing the behavior of during motion of the system are obtained by substituting (32) into the homogeneous part of (32). Thus, we have

After substituting Q1, Q2 and Q3 into (33) and extracting those terms, which contribute to the variational equation to *ϵ (0),* one obtains

However, the variational equations are obtained by setting the coefficients of and to zero. Thus, after integrating the results, we obtains respectively

And

Where

Therefore, when the effect of the moving mass is considered, the first approximation to the homogeneous system is given as

Where

is called the modified natural frequency representing the frequency of the system due to the presence of the moving mass. It is observed that when *∈ = 0*, we recover the frequency of the moving force problem. Thus, the homogeneous part of equation (30) can now be written as

Hence, the entire equation (25) takes the form

Eq. (40) is a prototype of (19). Thus, using the same procedure as in the previous section, the solution to (40) can be obtained and on inversion yields

Equation (41) represent the transverse displacement of the non-uniformly prestressed tapered beam with exponentially varying thickness resting on Vlasov foundation under variable harmonic moving mass.

# **Result and discussion**

The transversed displacement of a non-uniform elastic beam may increase without bound. Thus, one is interested in resonance condition. Eqs (24) and (41) clearly depicts that the non-uniformly prestressed tapered beam with exponentially varying thickness resting on Vlasov foundation will grow without bound whenever

From equation (38) we have

It can be deduce from eqs. (31) and (32) that

Therefore, for all values of m, that, with the same natural frequency, the critical speed for the system consisting of a simply supported non-uniformly prestressed tapered beam with exponentially varying thickness resting on Vlasov foundation and traversed by moving distributed force with uniform speed is greater than that of moving distributed mass problem. Thus, for the same natural frequency of the Timoshenko beam, resonance is attained earlier in the moving distributed mass system than in the moving distributed force system.

For the purpose of numerical analysis of the forgoing problem, the velocity of the moving load and the length of non-uniform elastic beam are 30m/s and 15m respectively. Furthermore,

. The deflection profile of a non-uniformly prestressed tapered beam with exponentially varying thickness traversed by a moving force is shown in fig (2a) – fig (2c). It is observed that as the values of the varying parameters are increased, the response amplitudes decrease. The same effect is shown for the moving mass model which are shown in fig (3a) - fig (3c). Figure (4) illustrates the response of the beam for moving force and moving mass. Clearly, the response amplitude due to the moving mass is less than that due to moving force. Thus, the moving force solution is not always an upper bound for the accurate solution for the beam problem.

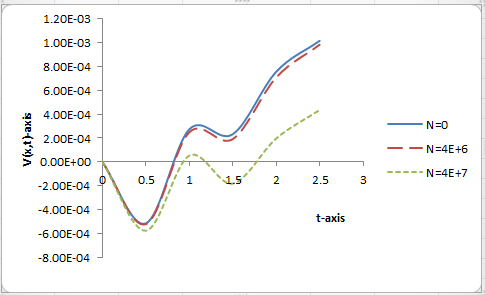


Fig. (2a): *Transverse displacement of the non-uniformly prestressed tapered beam with exponentially varying thickness resting on Vlasov foundation and traversed by moving distributed force for various value of axial force N and for fixed values of K1(4000) and K2(4000)*

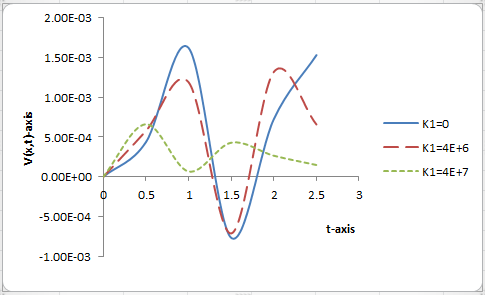


Fig. (2b): *Transverse displacement of the non-uniformly prestressed tapered beam with exponentially varying thickness resting on Vlasov foundation and traversed by moving distributed force for various value of foundation stiffness K1 and for fixed values of N (4000) and K2 (4000)*

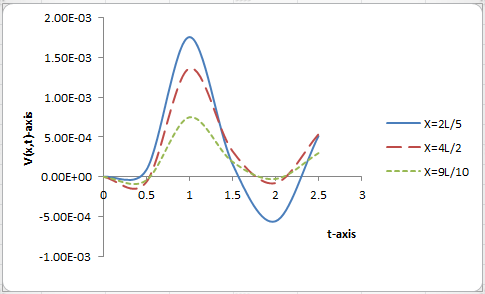


Fig. (2c): *Transverse displacement of the non-uniformly prestressed tapered beam with exponentially varying thickness resting on Vlasov foundation and traversed by moving distributed force for various value of distance X and for fixed values of N (4000) and K2 (4000)*

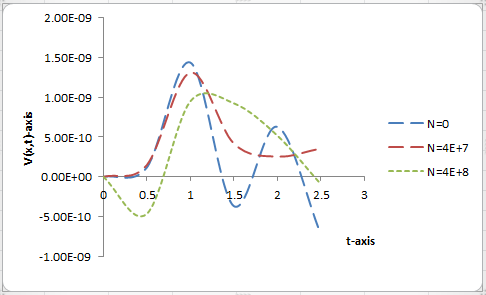


Fig. (3a): *Transverse displacement of the non-uniformly prestressed tapered beam with exponentially varying thickness resting on Vlasov foundation and traversed by moving distributed mass for various value of axial force N and for fixed values of K1(4000) and K2(4000)*

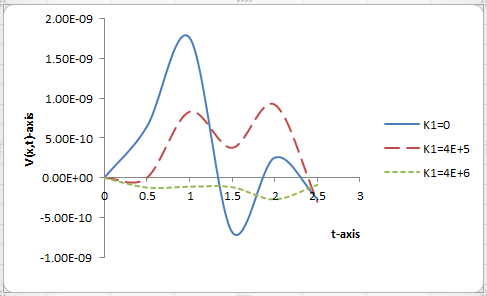


Fig. (3b): *Transverse displacement of the non-uniformly prestressed tapered beam with exponentially varying thickness resting on Vlasov foundation and traversed by moving distributed mass for various value of foundation stiffness K1 and for fixed values of N(4000) and K2(4000)*

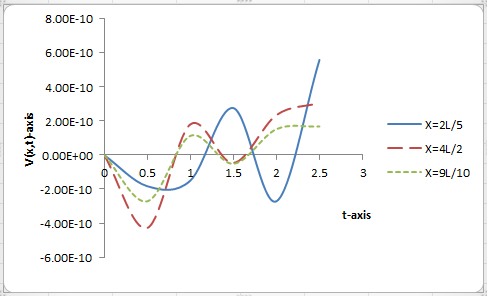


Fig. (3c): *Transverse displacement of the non-uniformly prestressed tapered beam with exponentially varying thickness resting on Vlasov foundation and traversed by moving distributed force for various value of distance X and for fixed values of N (4000) and K2 (4000)*

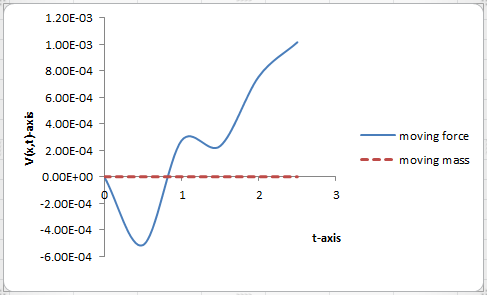


Fig. (4): *Comparison of the displacement response of the moving distributed force and moving distributed mass for the dynamical problem*

# **Conclusion**

An analytical solution is presented for the deflection response of non-uniformly prestressed tapered beam with exponentially varying thickness resting on Vlasov foundation under variable harmonic load moving with constant velocity. The solution technique is based on Galerkin's method and modification of the asymptotic method. The analysis exhibited the following features:

* The critical speeds of the system increases with an increase in the values of foundation stiffness, foundation modulus and exponential factor in the problem of non-prismatic Bernoulli-Euler beam with exponentially thickness resting on variable bi-parametric foundation.
* As the foundation parameters increased, the transverse deflection of the beam model decreased.
* It shows that inertia effects of a moving load must considered when heavy loads are involved.
* It showing that moving distributed force solution is not always an upper bound to the solution of a moving distributed mass problem.

Thus, the risk of resonance was reduced as the value of the varying parameters increased

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