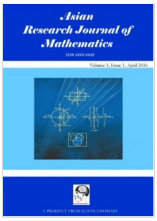


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# Dynamic Response of Non-uniformly Prestressed Thick Beam under Distributed Moving Load Travelling at Varying Velocity

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***Authors' contributions***

*This work was carried out in collaboration between all authors. Author SAJ designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of themanuscript. Authors SAJ, OKO and AOO managed the analysis ofthe study. Authors OKO and OOA managed the literature searches.All authors read and approved the final manuscript.*

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## Abstract

The dynamic response of non-uniformly prestressed thick beam under distributed moving load travelling at varying velocity is investigated in this paper. In order to obtain solution to the dynamical problem, a technique based on the method of Galerkin with the series representation of Heaviside function, was first used to transform the equation and thereafter the transformed equations were solved using Strubles asymptotic method and Laplace transformation techniques in conjunction with convolution theory.The displacement response for moving distributed force and moving distributed mass models for the dynamical problem are calculated for various time t and presented in plotted curves.

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Foremost, it is found that, the moving distributed force is not an upper bound for the accurate solution of the moving distributed mass problem, which shows that the inertia term must be considered for accurate assessment of the response to moving distributed load of elastic structural members . Analyses further shows that increase in the values of the structural parameters such as axial force N, shear modulus G and foundation stiffness K reduces the response amplitudes of non-uniformly prestressed thick beam under moving distributed loads. In order to verify the accuracy of the present method, the dynamic responses of a simply supported Timoshenko beam obtained by the present method and the frequency-domain spectral element method (SEM) are compared at two different velocities. The results shows that the dynamic responses obtained by the present method are almost identical to those obtained by using the SEM. Finally, for the same natural frequency, the critical speed for the beam transversed by moving distributed force is greater than that under the influence of a moving distributed mass. Hence resonance is reached earlier in the moving distributed mass problem.

*Keywords:* *Non-uniformly prestressed; varying velocity; strubble's asymptotic method; Galerkin's method; resonance.*

# 1 Introduction

The analysis of interaction between continuous elastic system and subsystem (load) has been a fundamental problem in structural dynamics for more than a century. However, it is especially in bridge engineering that this problem finds its widest field of application. This study helps to recognize when the structure is approaching an overstressed condition. In the history of elastic system-subsystem interaction, theoretically, the problem of moving load was first tackled by Willis [1]. In his research work, he considered the case in which the structural mass was considered smaller than the mass of the load and obtained an approximate solution to the problem. In the analysis of the effects of vehicles moving over large-span bridges, Inglis [2] developed a theory according to where the gravitational effects of the moving loads may be separated from the inertia ones. In the analysis, the force is considered as moving along the beam while the mass of the vehicles acts at a definite constant point, say $x_o$. The inertia action of the mass in the deformed structure is described by the D'Alembert's principle as a product of mass and acceleration. When the inertia effect of the moving load is considered, the governing differential equation of motion becomes complex and cumbersome and no longer has constant coefficients. In particular, the coefficients become variable and singular. If the inertia effect of the moving load is neglected, the problem is termed moving force problem and when it is retained, it is termed moving mass problem. The approximation model in which the vehicle-truck interaction is completely neglected has been described by [3] as the crudest approximation known to the literature of assessing the dynamic response of an elastic system which supports moving concentrated masses. It is assumed that these concentrated loads act at a point on the structure and along a single line in space as they move. That is, the moving load is modelled as a lumped load. However, in practice, it is well known that loads are actually distributed over a small segment or over the entire length of the structural member they traverse. When the moving load is distributed, the problem of investigating the load-structure interaction becomes much more complicated. Concentrated forces are mere mathematical idealization, and cannot be found in the real world, where forces are either body forces acting within the bulk of the material or within the volume. Among several authors who have worked extensively in this area of study are [4] who studied the dynamic deflection to non-uniform Rayleigh beam when under the action of distributed load. [5] studied the dynamic behaviour under moving distributed masses of non-uniform Rayleigh beam with general boundary conditions. [6] studied the motion of non-uniformly prestressed tapered beams with exponentially varying thickness resting on Vlasov foundation under variable harmonic load moving with constant velocity. However, in all the aforementioned works, Bernoulli-Euler and

Rayleigh beams model are often employed. Until recently, the effects of shear deformation and rotatory inertia on the dynamic response of Timoshenko beam were rarely discussed. The problem of thick beams under the action of a variable travelling transverse load was studied by [7] and in his study, he found that the transverse response of a deep beam decreases as the moving load frequency increases. [8] Studied the problem of vibration of multi-span Timoshenko beam. His study shows that the effects of rotatory inertia and shear deformation cause the modal frequencies of the Timoshenko beam to be less than those of Bernoulli-Euler beam. [9] Studied the dynamic response of a uniform deep beam resting on a Winkler elastic foundation and excited by a moving load. However, it is remarked at this juncture that in most of the existing literature in dynamics of structure under moving loads, effect of axial force on a dynamic system, method of solution have been restricted to numerical simulating [10, 11]. In the same vein, the problem of assessing the dynamic response of elastic structures carrying moving loads has so far received scanty attention in the literature for moving loads at non-uniform velocities. The more practical cases when the velocities at which these loads travel are no longer constants, but vary with time have received very little attention in the literature survey [12]. This is as a result of the complex space-time dependencies inherent in such problems. Specifically, even when the inertia effects of the moving load is neglected, analytical solutions involving integral transformation are both intractable and cumbersome.

Moreover, it is remarked at this juncture that practical problems such as acceleration and breaking of automobile on roadways and highway bridges, taking off and landing of air-crafts on runway and breaking and acceleration forces in the calculation of rails and railway bridges in which the motion is not uniform, but a function of time have intensified the need for the study of behaviour of structures under the action of loads moving with variable velocities. Consequently, this study investigates the dynamic response of non-uniformly prestressed of thick beam under distributed moving load travelling at varying velocities using analytical method as a method of solution.

## 2 Theory and Formulation of The Problem

In this study, the problem of a non-uniformly prestressed elastic structure and carrying a mass $M$ is investigated. The beam's properties such as moment of inertia $I$ and the mass per unit length $\mu$ of the beam remained constant along the span of length $L$. The transverse displacement $V(x, t)$ and angular displacement $\varphi(x, t)$ of the beam travelling at a non-uniform velocity as shown in fig (1# a and 1# b) is given as [13]

$$\frac{\partial}{\partial x}\left[K^{'}GA\left(\varphi(x,t)-\frac{\partial V(x,t)}{\partial x}\right)\right]+\mu\frac{\partial^2 V(x,t)}{\partial t^2}+KV(x,t)-G\frac{\partial^2 V(x,t)}{\partial x^2}-\frac{\partial}{\partial x}\left[F(x)\frac{\partial V(x,t)}{\partial x}\right]=q(x,t) \tag{2.1}$$

and

$$\frac{\partial}{\partial x}\left[EI\frac{\partial\varphi(x,t)}{\partial x}\right]+K^{'}GA\left[-\frac{\partial V(x,t)}{\partial x}-\varphi(x,t)\right]-I\rho\frac{\partial^2\varphi(x,t)}{\partial t^2}=0 \tag{2.2}$$

where, g, denotes the acceleration due to gravity, $\frac{d^2}{dt^2}$ is a convective acceleration operator, $\frac{\partial^2}{\partial t^2}$ is the support beam's acceleration at the point of contact with the moving mass, $\frac{df(t)}{dt}\frac{\partial^2}{\partial x\partial t}$, is the well-known coriolis acceleration, $\left(\frac{df(t)}{dt}\right)^2\frac{\partial^2}{\partial x^2}$, is the centripetal acceleration of the moving mass and $\frac{d^2f(t)}{dt^2}\frac{\partial}{\partial x}$, is the acceleration component in the vertical direction when the moving load is not constant.

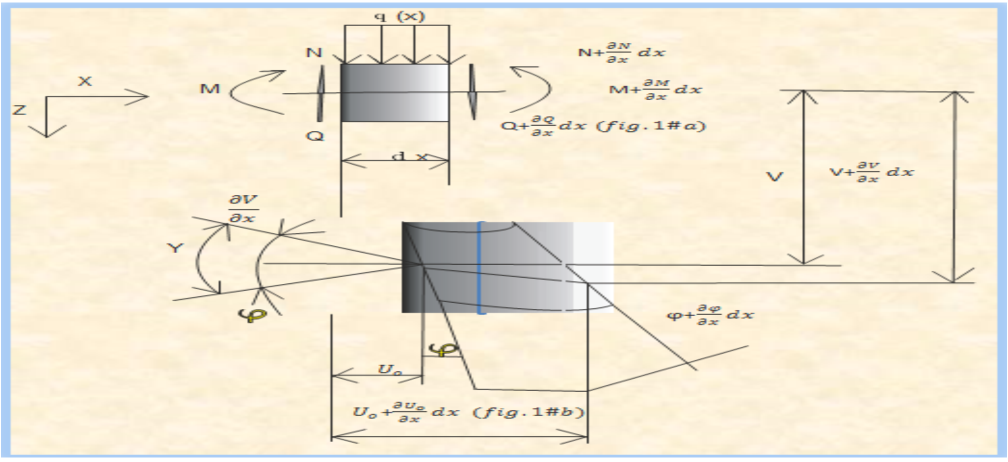


**Fig. 1#.** ***Geometry of a variable axial force on the deflection of thick beam.***

where, *t,* is the time coordinate, $\rho$, is the density of the beam, $A$, is the cross sectional area, $EI$, is the flexural stiffness, $x$, is the spatial coordinate, $K$, is the foundation stiffness, $G$, is the shear modulus, *F(x)*, is the non-uniform axial force, $K^{'}$, is the constant dependent on the shape of the beam and *q(x, t)*, is the uniform distributed load acting on the beam. In this problem, the distributed load moving on the beam under consideration has mass commensurable with the mass of the beam. Consequently, the load inertia is not negligible but significantly affects the behaviour of the dynamical system. Thus, the distributed load *q(x, t)* takes the form

$$q(x,t) = MgH\left[x - f\left(t\right)\right]\left[1 - \frac{1}{g}\frac{d^2V(x,t)}{dt^2}\right]; \qquad \frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2\frac{df(t)}{dt}\frac{\partial^2}{\partial x \partial t} + \left(\frac{df(t)}{dt}\right)^2\frac{\partial^2}{\partial x^2} + \frac{d^2f(t)}{dt^2}\frac{\partial}{\partial x} \tag{2.3}$$

In the same vein, the direction/distance travelled by the load on the beam at any given instance of the time and the prestressed terms are given as

$$f\left(t\right) = x_o + ct + \frac{1}{2}at^2; \quad F\left(x\right) = N_o\left(1 + sin\frac{\pi x}{L}\right) \tag{2.4}$$

where, $x_o$, represents the point of application of force $P_f$ *(x, t)* at any instant time $t = 0$, $c$, is the initial velocity and $a$, is the acceleration of the motion and $N_0$ is the constant axial force.

Moreover, the moving load is assumed to be of mass, $M$, and time, $t$, is assumed to be limited to that interval of time within which the mass $M$ is on the beam. i.e.

$$0 \leq f(t) \leq L \tag{2.5}$$

and $H\left[x - f(t)\right]$ is the Heaviside function, which is a typical engineering function made to measure engineering application involving function that are either "on" or "off" and it is defined as

$$H(x) = \begin{cases} 1, & x > ct. \\ 0, & x \leq ct. \end{cases} \qquad H\left[x - f(t)\right] = \begin{cases} 1, & x \geq f(t). \\ 0, & x < f(t). \end{cases} \tag{2.6}$$

It is remarked at this juncture that the beam under consideration is assumed to have simply supported ends beam at *x=0* and *x=L*. Thus the boundary conditions are

$$V\left(0,t\right) = V\left(L,t\right) = 0; \qquad \frac{\partial\varphi(0,t)}{\partial x} = \frac{\partial\varphi(L,t)}{\partial x} = 0 \tag{2.7}$$

The initial conditions however without any loss of generality is given as

$$V\left(x,0\right)=\frac{\partial V(x,0)}{\partial t}=0;\qquad \varphi\left(x,0\right)=\frac{\partial \varphi(x,0)}{\partial t}=0 \tag{2.8}$$

Therefore, substituting Eqs. (2.3) and (2.4) into (2.1) and (2.2), we obtains

$$\mu\frac{\partial^2 V}{\partial t^2}-K^{'}GA\left(\frac{\partial^2 V}{\partial x^2}-\frac{\partial \varphi}{\partial x}\right)-N_o\left(1+sin\frac{\pi x}{L}\right)\frac{\partial^2 V}{\partial x^2}-\frac{N_o\pi}{L}cos\frac{\pi x}{L}\frac{\partial V}{\partial x}+\ K\ V\left(x,t\right)-\ G\frac{\partial^2 V}{\partial x^2}$$
$$+MH\left[x-\left(x_o+ct+\frac{1}{2}at^2\right)\right]\left[\frac{\partial^2 V}{\partial t^2}+2\left(c+at\right)\frac{\partial^2}{\partial x\partial t}+\left(c+at\right)^2\frac{\partial^2 V}{\partial x^2}+a\frac{\partial V}{\partial x}\right]=MgH\left[x-\left(x_o+ct+\frac{1}{2}at^2\right)\right] \tag{2.9}$$

$$and\qquad \rho I\frac{\partial^2 \varphi}{\partial t^2}-K^{'}GA\left(\frac{\partial V}{\partial x}-\varphi\right)-EI\frac{\partial^2 \varphi}{\partial x^2}=0 \tag{2.10}$$

# 3 Solution Procedure

Unlike cases where axial force is constant, finite integral transform is inapplicable and we resort to a modification of the approximate analytical best suited for solving diverse problem in dynamics of structures generally referred to as *Galerkin's method*. Thus, we use the *Galerkin's method* described in [14, 15] to reduce the simultaneous second order partial differential equations to a sequence of simultaneous second order ordinary differential equations. Thus, a solution of the form

$$V(x,t)=\sum_{m=1}^{n}Y_m(t)U_m(x);\qquad \varphi\left(x,t\right)=\sum_{m=1}^{n}Z_m\left(t\right)X_m(x) \tag{3.1}$$

are sought, where $U_m(x)\ =\ sin\frac{m\pi x}{L}$ and $X_m(x)\ =\ cos\frac{m\pi x}{L}$ are chosen as a suitable kernel of the Galerkin's method in Eqs. (2.9) and (2.10) for simply supported boundary condition, due to displacement and rotation respectively. Therefore, substituting Eq. (3.1) into Eqs. (2.9) and (2.10) respectively, yields

$$\sum_{m=1}^{n}\Bigg\{\mu U_m(x)\ddot{Y}_m(t)-K^{'}GA\left[U_m^{"}\left(x\right)Y_m(t)-X_m^{'}(x)Z_m(t)\right]$$
$$-N_o\left[U_m^{"}(x)+sin\frac{m\pi x}{L}U_m^{"}(x)+\frac{\pi}{L}cos\frac{m\pi x}{L}U_m^{'}(x)\right]Y_m(t)+\left[KU_m(x)-GU_m^{"}(x)\right]Y_m(t)$$
$$+MH\left[x-\left(x_o+ct+\frac{1}{2}at^2\right)\right]\left[U_m(x)\ddot{Y}_m\left(t\right)+2(c+at)U_m^{'}(x)\dot{Y}_m(t)+(c+at)^2U_m^{"}(x)\,Y_m(t)\right.$$
$$\left.+aU_m^{'}(x)Y_m(t)\right]-\left[x-(x_o+ct+\frac{1}{2}at^2)\right]\Bigg\}=0 \tag{3.2}$$

$$and\qquad \sum_{m=1}^{n}\left[\rho IX_m\left(x\right)\ddot{Z}_m\left(t\right)+K^{'}GAX_m\left(x\right)Z_m\left(t\right)-EIU_m^{"}\left(x\right)Z_m\left(t\right)-K^{'}GAU_m^{'}\left(x\right)Y_m\left(t\right)\right]=0 \tag{3.3}$$

In order to determine an expression for $Y_m(t)$ and $Z_m(t)$, it is required that the expression on the left hand side of Eqs. (3.2) and (3.3) are orthogonal to the functions $sin\frac{k\pi x}{L}$ and $cos\frac{k\pi x}{L}$ respectively.

Thus, we have

$$R_1(m,k)\ddot{Y}_m(t) + R_2(m,k)Y_m(t) + R_3(m,k)Z_m(t) + \frac{M}{\mu}\Big[R_4(m,k)\ddot{Y}_m t(t) + 2(c+at)R_5(m,k)\dot{Y}_m(t) + (c+at)^2 R_6(m,k)Y_m(t) + aR_7(m,k)Y_m(t)\Big] = \frac{Mg}{\mu}R_8(k) \quad (3.4)$$

*and* $$S_1\left(m,k\right)\ddot{Z}_m\left(t\right) + S_2\left(m,k\right)Z_m\left(t\right) - S_3\left(m,k\right)Y_m\left(t\right) = 0 \quad (3.5)$$

where

$$R_1\left(m,k\right) = I_1 = \int_0^L sin\frac{m\pi x}{L}sin\frac{k\pi x}{L}\ dx;\quad R_2\left(m,k\right) = R_{2a}\left(m,k\right) - R_{2b}\left(m,k\right) - R_{2c}\left(m,k\right) - R_{2d}\left(m,k\right);$$

$$-R_{2e}\left(m,k\right) - R_{2f}\left(m,k\right);\quad R_{2a}\left(m,k\right) = \frac{K}{\mu}I_1;\quad R_{2b}\left(m,k\right) = -\frac{m^2\pi^2K'GA}{\mu L^2}I_1;\quad R_{2c}\left(m,k\right) = -\frac{m^2\pi^2N_o}{\mu L^2}I_1$$

$$R_{2d}\left(m,k\right) = -\frac{m^2\pi^2N_o}{\mu L^2}I_2;\quad R_{2e}\left(m,k\right) = \frac{m\pi^2N_o}{\mu L^2}I_3;\quad R_{2f}\left(m,k\right) = -\frac{m^2\pi^2G}{\mu L^2}I_1;\quad R_3\left(m,k\right) = -\frac{m\pi K'GA}{\mu L}I_1$$

$$R_4\left(m,k\right) = I_4;\quad R_5\left(m,k\right) = \frac{m\pi}{L}I_5;\quad R_6\left(m,k\right) = -\frac{m^2\pi^2}{L^2}I_4;\quad R_7\left(m,k\right) = \frac{m\pi}{L}I_5;\quad R_8\left(k\right) = I_6;\quad S_1\left(m,k\right) = I_7;$$

$$S_2\left(m,k\right) = S_{2a}\left(m,k\right) - S_{2b}\left(m,k\right)\quad S_{2a}\left(m,k\right) = \frac{K'GA}{\rho I}I_7;\quad S_{2b}\left(m,k\right) = -\frac{m^2\pi^2E}{\rho L^2}I_7;$$

$$S_3\left(m,k\right) = \frac{m\pi K'GA}{\rho IL}I_7\quad I_2 = \int_0^L sin\frac{\pi x}{L}sin\frac{m\pi x}{L}sin\frac{k\pi x}{L}\ dx;\quad I_3 = \int_0^L cos\frac{\pi x}{L}cos\frac{m\pi x}{L}sin\frac{k\pi x}{L}\ dx$$

$$I_4 = \int_0^L H\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]sin\frac{m\pi x}{L}sin\frac{k\pi x}{L}\ dx\quad I_5 = \int_0^L H\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]cos\frac{m\pi x}{L}sin\frac{k\pi x}{L}\ dx$$

$$I_6 = \int_0^L H\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]sin\frac{k\pi x}{L}\ dx;\quad I_7 = \int_0^L cos\frac{m\pi x}{L}cos\frac{k\pi x}{L}\ dx \quad (3.6)$$

Using the property of Heaviside function, it can be expressed in series form given by [14] i.e.

$$H\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right] = \frac{1}{4} + \frac{1}{\pi}\sum_{n=0}^{\infty}\sin\left(2n+1\right)\ \pi x\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right] \quad (3.7)$$

Thus, in view of (3.6) and (3.7), it can be shown that

$$R_1\left(m,k\right)\ddot{Y}_m\left(t\right) + R_2\left(m,k\right)Y_m\left(t\right) + R_3\left(m,k\right)Z_m\left(t\right) + \frac{M}{\mu}\Bigg\{\Bigg[\frac{I_1}{4} + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos\left(2n+1\right)\ \pi\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]}{2n+1}\ I_{8a}$$

$$-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left(2n+1\right)\ \pi\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]}{2n+1}I_{8b}\Bigg]\ddot{Y}_m(t) + 2\left(c+at\right)\Bigg[\frac{I_9}{4} + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos\left(2n+1\right)\ \pi\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]}{2n+1}\ I_{9a}$$

$$-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left(2n+1\right)\ \pi\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]}{2n+1}I_{9b}\Bigg]\dot{Y}_m(t) - \frac{m^2\pi^2}{L^2}\left(c+at\right)^2\Bigg[\frac{I_1}{4} + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos\left(2n+1\right)\ \pi\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]}{2n+1}\ I_{8a}$$

$$-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left(2n+1\right)\ \pi\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]}{2n+1}I_{8b}\Bigg]Y_m(t) + \frac{m\pi}{L}a\Bigg[\frac{I_9}{4} + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos\left(2n+1\right)\ \pi\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]}{2n+1}\ I_{9a}$$

$$-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left(2n+1\right)\ \pi\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]}{2n+1}I_{9b}\Bigg]Y_m\left(t\right)\Bigg\} = \frac{Mg}{\mu}I_6 \quad (3.8)$$

*and* $$S_1\left(m,k\right)\ddot{Z}_m\left(t\right) + S_2\left(m,k\right)Z_m\left(t\right) - S_3\left(m,k\right)Y_m\left(t\right) = 0 \quad (3.9)$$

*where* $$I_{8a} = \int_0^L \sin(2n+1) sin\frac{m\pi x}{L} sin\frac{k\pi x}{L}\; dx; \quad I_{8b} = \int_0^L \cos(2n+1) sin\frac{m\pi x}{L} sin\frac{k\pi x}{L}\; dx$$
$$I_9 = \int_0^L cos\frac{m\pi x}{L} sin\frac{k\pi x}{L}\; dx; \quad I_{9a} = \int_0^L \sin(2n+1) cos\frac{m\pi x}{L} sin\frac{k\pi x}{L}\; dx;$$
$$I_{9b} = \int_0^L \cos(2n+1) cos\frac{m\pi x}{L} sin\frac{k\pi x}{L}\; dx \tag{3.10}$$

Evaluating the integrals $I_1$ to $I_{9b}$, thereafter substituting the results of the integrals into Eqs. (3.8) and(3.9), yields

$$\ddot{Y}_m(t) + \propto_1{}^2 Y_m(t) + Q_1 Z_m(t) \quad + \varepsilon_o \Bigg\{ \Bigg[ \frac{L}{4} + \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{\cos(2n+1)\,\pi\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]}{2n+1}\; I_{8a}$$
$$-\frac{1}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)\,\pi\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]}{2n+1} I_{8b}\Bigg]\ddot{Y}_m(t) + 2(c+at)\Bigg[-\frac{KL}{\pi(m^2-k^2)}$$
$$+\frac{2}{\pi}\sum_{n=0}^{\infty} \frac{\cos(2n+1)\,\pi\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]}{2n+1}\; I_{9a} - \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)\,\pi\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]}{2n+1} I_{9b}\Bigg]\dot{Y}_m(t)$$
$$-\frac{m^2\pi^2}{L^2}(c+at)^2\Bigg[\frac{L}{4} + \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{\cos(2n+1)\,\pi\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]}{2n+1}\; I_{8a}$$
$$-\frac{2}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)\,\pi\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]}{2n+1} I_{8b}\Bigg] Y_m(t) + \frac{m\pi}{L} a\Bigg[-\frac{KL}{\pi(m^2-k^2)}$$
$$+\frac{2}{\pi}\sum_{n=0}^{\infty} \frac{\cos(2n+1)\,\pi\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]}{2n+1}\; I_{9a}$$
$$-\frac{2}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)\,\pi\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]}{2n+1} I_{9b}\Bigg] Y_m(t)\Bigg\} = \frac{2Mg}{\pi k\mu}\left[cos\frac{\pi k\left(x_o + ct + \frac{1}{2}at^2\right)}{L} - (-1)^{-1}\right] \tag{3.11}$$

$$\textit{and} \qquad \ddot{Z}_m(t) + \propto_2{}^2 Z_m(t) - Q_2 Y_m(t) = 0 \tag{3.12}$$

where

$$\propto_1{}^2 = \frac{R_2(m,k)}{R_1(m,k)}; \quad \propto_2{}^2 = \frac{S_2(m,k)}{S_1(m,k)}; \quad Q_1 = \frac{R_3(m,k)}{R_1(m,k)}; \quad Q_2 = \frac{S_3(m,k)}{S_1(m,k)}; \quad \varepsilon_o = \frac{M}{\mu L} \tag{3.13}$$

Eqs. (3.11) and (3.12) are now the fundamental equations governing the dynamical problem of the non-uniformly prestressed thick beam under distributed moving load travelling at varying velocity. It follows that two special cases of (3.11) arises, namely *Moving Force* and *Moving Mass* problem.

## 3.1 Non-uniformly Prestressed Thick Beam Traversed by Moving Force

In this section, an approximate model of the differential equation describing the response of the elastic structure is obtained by neglecting inertia terms. i.e. $\varepsilon_o$=0. Thus, after some simplification by writing acceleration '$a$' in terms of velocity, thereafter using trigonometry identity Eqs. (3.11) and (3.12) are of the form

$$\ddot{Y}_m(t) + \propto_{f1}{}^2 Y_m(t) + Q_1 Z_m(t) = P_o\left[C_{nx}\cos\theta t \; - S_{nx}\sin\theta t \; - R_k\right] \tag{3.14}$$

$$\textit{and} \qquad \ddot{Z}_m(t) + \propto_{f2}{}^2 Z_m(t) - Q_2 Y_m(t) = 0 \tag{3.15}$$

$$where \quad {\propto_{f1}}^2 = {\propto_1}^2; \quad {\propto_{f2}}^2 = {\propto_2}^2; \quad P_o = \frac{2Mg}{\pi k\mu}; \quad n_o = \frac{k}{L}; \quad R_k = \ (-1)^{-1} \quad C_{nx} = \cos n_o\pi \ x_o;$$
$$S_{nx} = \sin n_o\pi \ x_o; \quad \theta = \ n_o U_o; \quad U_o = \frac{1}{2}(u + c) \tag{3.16}$$

Where $U_o$ is the average velocity at any point and $u$ is the velocity at any instant time $t$. Subjecting Eqs. (3.14) and (3.15) to a Laplace transform in conjunction with the initial boundary condition defined in (2.8), one obtains simple algebraic simultaneous equations define as

$$\overline{Y}_m(s)\left(s^2 + {\propto_{f1}}^2\right) + Q_1\overline{Z}_m(s) = P_o\left[C_{nx}\frac{s}{s^2+\theta^2} - S_{nx}\frac{\theta}{s^2+\theta^2} - \frac{R_k}{s}\right] \tag{3.17}$$

$$and \quad \overline{Z}_m(s)\left(s^2 + {\propto_{f1}}^2\right) - Q_2\overline{Y}_m(s) = 0 \tag{3.18}$$

Solving Eqs. (3.17) and (3.18) simultaneously, yields

$$\overline{Z}_m(s) = \frac{P_oQ_2}{{\omega_{f1}}^2 - {\omega_{f2}}^2}\left(\frac{1}{s^2+{\omega_{f2}}^2} - \frac{1}{s^2+{\omega_{f2}}^2}\right)\left(C_{nx}\frac{s}{s^2+\theta^2} - S_{nx}\frac{\theta}{s^2+\theta^2} - \frac{R_k}{s}\right) \tag{3.19}$$

and

$$\overline{Y}_m(s) = \frac{P_o}{{\omega_{f1}}^2 - {\omega_{f2}}^2}\left(\frac{{\propto_{f2}}^2 - {\omega_{f2}}^2}{s^2+{\omega_{f2}}^2} - \frac{{\propto_{f2}}^2 - {\omega_{f1}}^2}{s^2+{\omega_{f2}}^2}\right)\left(C_{nx}\frac{s}{s^2+\theta^2} - S_{nx}\frac{\theta}{s^2+\theta^2} - \frac{R_k}{s}\right) \tag{3.20}$$

where

$$\begin{aligned} {\omega_{f1}}^2 &= \frac{1}{2}\left[\left({\propto_{f1}}^2 + {\propto_{f2}}^2\right) - \sqrt{\left({\propto_{f1}}^2 - {\propto_{f2}}^2\right)^2 - 4Q_1Q_2}\right]; \\ {\omega_{f2}}^2 &= \frac{1}{2}\left[\left({\propto_{f1}}^2 + {\propto_{f2}}^2\right) + \sqrt{\left({\propto_{f1}}^2 - {\propto_{f2}}^2\right)^2 - 4Q_1Q_2}\right] \end{aligned} \tag{3.21}$$

Thus, Eqs. (3.19) and (3.20) reduce to that of finding Laplace inversion of $\overline{Z}_m(s)$ *and* $\overline{Y}_m(s)$. To do this, we adopt the following representations

$$\begin{aligned} \mathrm{g}(s) &= \left(C_{nx}\frac{s}{s^2+\theta^2} - S_{nx}\frac{\theta}{s^2+\grave{e}^2} - \frac{R_k}{s}\right); \quad \mathrm{f}_1(s) = \left(\frac{1}{s^2+{\omega_{f2}}^2} - \frac{1}{s^2+{\omega_{f2}}^2}\right); \\ \mathrm{f}_2(s) &= \left(\frac{{\propto_{f2}}^2 - {\omega_{f2}}^2}{s^2+{\omega_{f2}}^2} - \frac{{\propto_{f2}}^2 - {\omega_{f1}}^2}{s^2+{\omega_{f2}}^2}\right) \end{aligned} \tag{3.22}$$

So that the Laplace inversion of (3.19) and (3.20) is the convolution of $f_i(s)$ and $g(s)$ defined as

$$\mathrm{f}_i(s) = \int_0^t f_i(t-u)\,g(u)\,du \tag{3.23}$$

Thus, the Laplace inversion of the two equations respectively given as

$$Z_m(t) = \frac{P_oQ_2}{{\omega_{f1}}^2 - {\omega_{f2}}^2}\left[\frac{1}{\omega_{f2}}\left(C_{nx}I_{11} - S_{nx}I_{13} - R_kI_{15}\right) - \frac{1}{\omega_{f1}}\left(C_{nx}I_{12} - S_{nx}I_{14} - R_kI_{16}\right)\right] \tag{3.24}$$

and

$$\begin{aligned} Y_m(t) = \frac{P_o}{{\omega_{f1}}^2 - {\omega_{f2}}^2}\Bigg[&\frac{{\propto_{f2}}^2 - {\omega_{f2}}^2}{\omega_{f2}}\left(C_{nx}I_{11} - S_{nx}I_{13} - R_kI_{15}\right) \\ &- \frac{{\propto_{f2}}^2 - {\omega_{f1}}^2}{\omega_{f1}}\left(C_{nx}I_{12} - S_{nx}I_{14} - R_kI_{16}\right)\Bigg] \end{aligned} \tag{3.25}$$

where

$$\mathrm{I}_{11}=\int_0^t sin\omega_{f2}\left(t-u\right)cos\theta u\ du;\quad \mathrm{I}_{12}=\int_0^t sin\omega_{f1}\left(t-u\right)cos\theta u\ du\ \mathrm{I}_{13}=\int_0^t sin\omega_{f2}\left(t-u\right)sin\theta u\ du;$$
$$\mathrm{I}_{14}=\int_0^t sin\omega_{f1}\left(t-u\right)sin\theta u\ du\quad \mathrm{I}_{15}=\int_0^t sin\omega_{f2}\left(t-u\right)\ du;\quad \mathrm{I}_{16}=\int_0^t sin\omega_{f1}\left(t-u\right)\ du \tag{3.26}$$

Therefore, solving the integrals in (3.26) and then substitute the results into (3.24) and (3.25), after inversion using Eq. (3.1) one obtains

$$\varphi\left(x,t\right)=\sum_{m=1}^{n}\frac{P_oQ_2}{{\omega_{f1}}^2{\omega_{f2}}^2\left({\omega_{f1}}^2-{\omega_{f2}}^2\right)\left({\omega_{f1}}^2-\theta^2\right)\left({\omega_{f2}}^2-\theta^2\right)}\Bigg\{{\omega_{f1}}^2\left({\omega_{f1}}^2-\theta^2\right)\Big[C_{nx}\omega_{f2}\big(cos\theta t$$
$$-cos\omega_{f2}t\big)-S_{nx}\omega_{f2}\left(\omega_{f2}sin\theta t-\theta sin\omega_{f2}t\right)-R_k\left({\omega_{f2}}^2-\theta^2\right)\left(1-cos\omega_{f2}t\right)\Big]-{\omega_{f2}}^2\left({\omega_{f2}}^2-\theta^2\right)\times$$
$$\Big[C_{nx}\omega_{f1}\left(cos\theta t-cos\omega_{f1}t\right)-S_{nx}\omega_{f1}\left(\omega_{f1}sin\theta t-\theta sin\omega_{f1}t\right)-R_k\left({\omega_{f1}}^2-\theta^2\right)\left(1-cos\omega_{f1}t\right)\Big]\Bigg\}\ cos\frac{m\pi x}{L} \tag{3.27}$$

$$and\qquad V\left(x,t\right)=\sum_{m=1}^{n}\frac{P_oQ_2}{{\omega_{f1}}^2{\omega_{f2}}^2\left({\omega_{f1}}^2-{\omega_{f2}}^2\right)\left({\omega_{f1}}^2-\theta^2\right)\left({\omega_{f2}}^2-\theta^2\right)}\Bigg\{{\omega_{f1}}^2\left({\omega_{f1}}^2-\theta^2\right)\big({\propto_{f2}}^2$$
$$-\ {\omega_{f2}}^2\big)\times\Big[C_{nx}\omega_{f2}\left(cos\theta t-cos\omega_{f2}t\right)-S_{nx}\omega_{f2}\left(\omega_{f2}sin\theta t-\theta sin\omega_{f2}t\right)-R_k\left({\omega_{f2}}^2-\theta^2\right)\left(1-cos\omega_{f2}t\right)\Big]$$
$$-{\omega_{f2}}^2\left({\omega_{f2}}^2-\theta^2\right)\times\left({\propto_{f2}}^2-\ {\omega_{f1}}^2\right)\Big[C_{nx}\omega_{f1}\left(cos\theta t-cos\omega_{f1}t\right)-S_{nx}\omega_{f1}\left(\omega_{f1}sin\theta t-\theta sin\omega_{f1}t\right)$$
$$-R_k\left({\omega_{f1}}^2-\theta^2\right)\left(1-cos\omega_{f1}t\right)\Big]\Bigg\}\sin\frac{m\pi x}{L} \tag{3.28}$$

Eqs. (3.27) and (3.28) respectively represent the angular and transverse displacements of a non-uniformly prestressed thick beam under distributed moving force travelling at varying velocity.

## 3.2 Non-uniformly Prestressed Thick Beam Traversed by Moving Mass

In this section, the solution to the entire equation Eq. (3.11) is sought when no terms of the coupled differential equation is neglected. an approximate analytical solution to Eq. (3.11) is resort to. Thus, we used a modification of the asymptotic method due to Strubble's technique which is often used in treating oscillatory system. To this ends, equation (3.11) is rearranged to take the form

$$\ddot{Y}_m\left(t\right)+\frac{2\varepsilon_ouQ_2\left(n,m,k\right)}{1+\varepsilon_oQ_1\left(n,m,k\right)}\dot{Y}_m\left(t\right)+\frac{{\propto_{f1}}^2+{m_L}^2u^2\varepsilon_oQ_1\left(n,m,k\right)+am_L\varepsilon_oQ_2\left(n,m,k\right)}{1+\varepsilon_oQ_1\left(n,m,k\right)}$$
$$=\frac{P_o\left[cosn_o\pi\left(x_o+u_ot\right)-R_k\right]}{1+\varepsilon_oQ_1\left(n,m,k\right)} \tag{3.29}$$

where

$$Q_1(n,m,k)=\left[\frac{L}{4}+\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\,\pi(x_o+U_ot)}{2n+1}\,I_{8a}-\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\,\pi(x_o+U_ot)}{2n+1}I_{8b}\right];$$

$$Q_2(n,m,k)=\left[\frac{-kL}{\pi(m^2-k^2)}+\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\,\pi(x_o+U_ot)}{2n+1}\,I_{9a}-\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\,\pi(x_o+U_ot)}{2n+1}I_{9b}\right];\qquad(3.30)$$

$$m_L=\frac{m\pi}{L};\qquad a=\frac{u-c}{t}$$

With this technique, one seek the modified frequency corresponding to the frequency of the free system due to the presence of the effect of the moving mass. Following the procedures extensively discussed in [14, 15], the homogeneous part of equation (3.29) is simplified to take the form

$$\ddot{Y}_m(t)+{\propto_{m1}}^2Y_m(t)=0\qquad(3.31)$$

where

$$\propto_{m1}=\propto_{f1}\left[1-\frac{\in}{8}\left(\frac{4amk}{{\propto_{f1}}^2(m^2-k^2)}-\frac{(m\pi u)^2}{{\propto_{f1}}^2L}-L\right)\right]\qquad(3.32)$$

is called the modified natural frequency representing the frequency of the system due to the presence of the moving mass. Thus, the entire equation (3.11) and (3.12) reduces to

$$\ddot{Y}_m(t)+{\propto_{m1}}^2Y_m(t)+Q_1Z_m(t)=P_o\left[C_{nx}\cos\theta t\ -S_{nx}\sin\theta t\ -R_k\right]\qquad(3.33)$$

$$\textit{and}\qquad\ddot{Z}_m(t)+{\propto_2}^2Z_m(t)-Q_2Y_m(t)=0\qquad(3.34)$$

(3.33) and (3.34) together, these equations are analogous to equations (3.14) and (3.15). Thus, using the same procedure as in the previous section, one obtains

$$\varphi(x,t)=\sum_{m=1}^{n}\frac{P_oQ_2}{{\omega_{m1}}^2{\omega_{m2}}^2({\omega_{m1}}^2-{\omega_{m2}}^2)({\omega_{m1}}^2-\theta^2)({\omega_{m2}}^2-\theta^2)}\left\{{\omega_{m1}}^2\left({\omega_{m1}}^2-\theta^2\right)\times\right.$$
$$\left[C_{nx}\omega_{m2}\left(cos\theta t-cos\omega_{m2}t\right)-S_{nx}\omega_{m2}\left(\omega_{m2}sin\theta t-\theta sin\omega_{m2}t\right)-R_k\left({\omega_{m2}}^2-\theta^2\right)\left(1-cos\omega_{m2}t\right)\right]$$
$$-{\omega_{m2}}^2\left({\omega_{m2}}^2-\theta^2\right)\left[C_{nx}\omega_{m1}\left(cos\theta t-cos\omega_{m1}t\right)-S_{nx}\omega_{m1}\left(\omega_{m1}sin\theta t-\theta sin\omega_{m1}t\right)\right.$$
$$\left.\left.-R_k\left({\omega_{m1}}^2-\theta^2\right)\left(1-cos\omega_{m1}t\right)\right]\right\}cos\frac{m\pi x}{L}\qquad(3.35)$$

and

$$V(x,t)=\sum_{m=1}^{n}\frac{P_oQ_2}{{\omega_{m1}}^2{\omega_{m2}}^2({\omega_{m1}}^2-{\omega_{m2}}^2)({\omega_{m1}}^2-\theta^2)({\omega_{m2}}^2-\theta^2)}\left\{{\omega_{m1}}^2\left({\omega_{m1}}^2-\theta^2\right)\times\right.$$
$$\left({\propto_{f2}}^2-{\omega_{m2}}^2\right)\left[C_{nx}\omega_{m2}\left(cos\theta t-cos\omega_{m2}t\right)-S_{nx}\omega_{m2}\left(\omega_{m2}sin\theta t-\theta sin\omega_{m2}t\right)\right.$$
$$\left.-R_k\left({\omega_{m2}}^2-\theta^2\right)\left(1-cos\omega_{m2}t\right)\right]-{\omega_{m2}}^2\left({\omega_{m2}}^2-\theta^2\right)\left({\propto_{f2}}^2-{\omega_{m1}}^2\right)\left[C_{nx}\omega_{m1}\left(cos\theta t-cos\omega_{m1}t\right)\right.$$
$$\left.\left.-S_{nx}\omega_{m1}\left(\omega_{m1}sin\theta t-\theta sin\omega_{m1}t\right)-R_k\left({\omega_{m1}}^2-\theta^2\right)\left(1-cos\omega_{m1}t\right)\right]\right\}\sin\frac{m\pi x}{L}\qquad(3.36)$$

Eqs. (3.35) and (3.36) respectively represent the angular and transverse displacements of a non-uniformly prestressed thick beam under distributed moving mass travelling at varying velocity.

# 4 Comments On The Closed Form Solutions

Theoretically speaking, the deflections of the Thick beam may increase beyond bounds. Practically, this means that the beam is in a state of resonance. The speed of the load which brings about resonance effect in the system is termed the critical speed. Eq. (3.28) clearly shows that the simply supported non-uniformly prestressed thick beam and traversed by a moving distributed force reaches a state of resonance whenever

$$\omega_{f1} = \theta; \quad \omega_{f2} = \theta; \quad \omega_{f1} = \omega_{f2} \tag{4.1}$$

but $\theta = \frac{kU_o}{L}$, so that $U_{cr} = \omega_{f1}L/K$ is the critical speed of the moving force system.

Similarly, when

the same system is under a moving mass, equation (3.36) shows that the corresponding resonance condition is

$$\omega_{m1} = \theta; \quad \omega_{m2} = \theta; \quad \omega_{m1} = \omega_{m2} \tag{4.2}$$

Using Eq.(3.21), (3.32), (4.1), and (4.2), it easily shown that

$$\omega_{m1} = \omega_{f1}\left[1 - \frac{\in}{4}\left(\frac{4amk}{\propto_{f1}{}^2(m^2-k^2)} - \frac{(m\pi u)^2}{\propto_{f1}{}^2 L} - L\right)\left[1 + \frac{\propto_{f2}{}^2}{2\omega_{f1}{}^2} - \frac{\in}{16}\left(\frac{4amk}{\propto_{f1}{}^2(m^2-k^2)} - \frac{(m\pi u)^2}{\propto_{f1}{}^2 L} - L\right)\left(1 - \frac{\propto_{f2}{}^2}{2\omega_{f1}{}^2}\right)\right]\right]^{\frac{1}{2}} \tag{4.3}$$

and

$$\omega_{m2} = \omega_{f2}\left[1 - \frac{\in}{4}\left(\frac{4amk}{\propto_{f1}{}^2(m^2-k^2)} - \frac{(m\pi u)^2}{\propto_{f1}{}^2 L} - L\right)\left[1 + \frac{\propto_{f2}{}^2}{2\omega_{f2}{}^2} - \frac{\in}{16}\left(\frac{4amk}{\propto_{f1}{}^2(m^2-k^2)} - \frac{(m\pi u)^2}{\propto_{f1}{}^2 L} - L\right)\left(1 - \frac{\propto_{f2}{}^2}{2\omega_{f2}{}^2}\right)\right]\right]^{\frac{1}{2}} \tag{4.4}$$

since $\omega_{f1}\left[1 - \frac{\in}{4}\left(\frac{4amk}{\propto_{f1}{}^2(m^2-k^2)} - \frac{(m\pi u)^2}{\propto_{f1}{}^2 L} - L\right)\left[1 + \frac{\propto_{f2}{}^2}{2\omega_{f1}{}^2} - \frac{\in}{16}\left(\frac{4amk}{\propto_{f1}{}^2(m^2-k^2)} - \frac{(m\pi u)^2}{\propto_{f1}{}^2 L} - L\right)\left(1 - \frac{\propto_{f2}{}^2}{2\omega_{f1}{}^2}\right)\right]\right]^{\frac{1}{2}}$

and $\omega_{f2}\left[1 - \frac{\in}{4}\left(\frac{4amk}{\propto_{f1}{}^2(m^2-k^2)} - \frac{(m\pi u)^2}{\propto_{f1}{}^2 L} - L\right)\left[1 + \frac{\propto_{f2}{}^2}{2\omega_{f2}{}^2} - \frac{\in}{16}\left(\frac{4amk}{\propto_{f1}{}^2(m^2-k^2)} - \frac{(m\pi u)^2}{\propto_{f1}{}^2 L} - L\right)\left(1 - \frac{\propto_{f2}{}^2}{2\omega_{f2}{}^2}\right)\right]\right]^{\frac{1}{2}}$

are less than one for all m, it can be deduce that, for the same natural frequency, the critical speed for the system consisting of a simply supported non-uniformly prestressed thick beam and traversed by moving distributed force with varying speed is greater than that of moving distributed mass problem. Thus, for the same natural frequency of the non-uniformly prestressed thick beam, resonance is attained earlier in the moving distributed mass system than in the moving distributed force system.

# 5 Analysis of Result and Discussion

In order to illustrate the analysis in view, the uniform beam of length $L$=17.5 m is considered. The load initial velocity $c$=30 ms$^{-1}$, Young modulus $E$=2.02×10$^{11}$ Nm$^{-3}$, moment of inertia $I$=.0012 m$^4$, $\pi$=22/7, cross sectional area $A$=7.175 m$^2$, density of the beam $\mu$=2400 kgm$^{-3}$, shear coefficient $K'$=5/6, shear modulus $G$=7.7E×10$^3$ Nm$^{-2}$, load's acceleration $a = 8ms^{-2}$ and the gravitational acceleration $g$=9.8 ms$^{-2}$ are used. The transverse displacement and angular displacement of the beam are calculated and plotted against time for various values of axial force $N$ with different values of length $L$ of the beam. The results are as shown on the various graphs below.

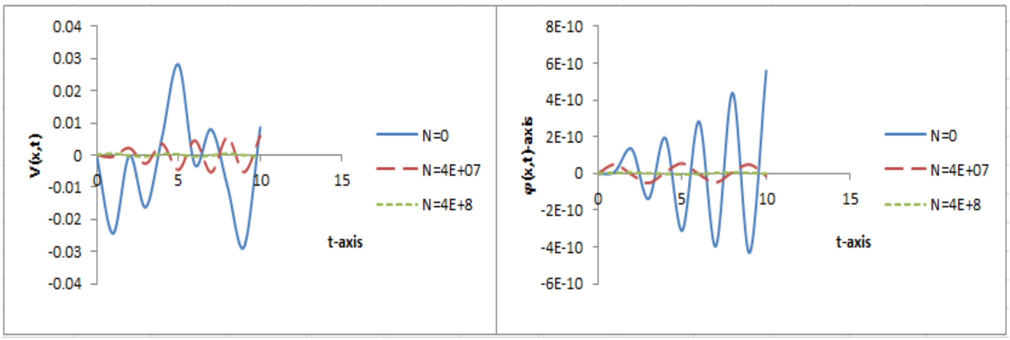


**Fig. 1. Transverse displacement & Rotation of the non-uniformly prestressed thick beam under distributed moving load travelling at varying velocity for various values of axial force N and for fixed values of G(77000) and K(40000) and traversed by moving distributed force.**

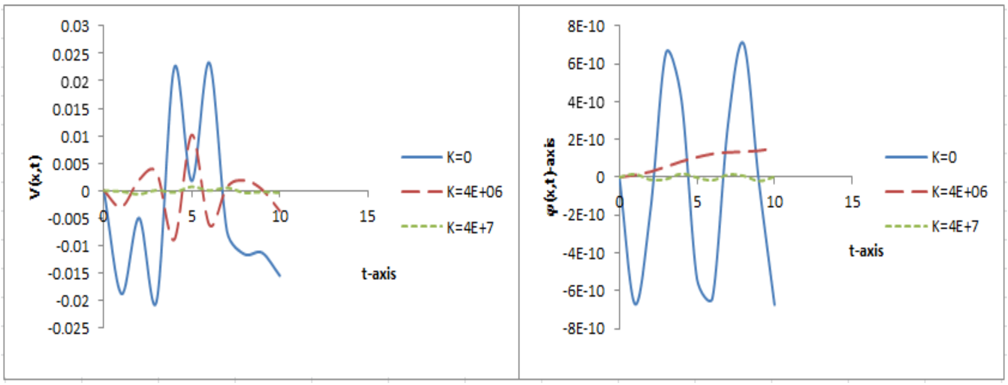


**Fig. 2. Transverse displacement & Rotation of the non-uniformly prestressed thick beam under distributed moving load travelling at varying velocity for various values of foundation stiffness K and fixed values of G(77000) and N(40000) that traversed by moving distributed force.**

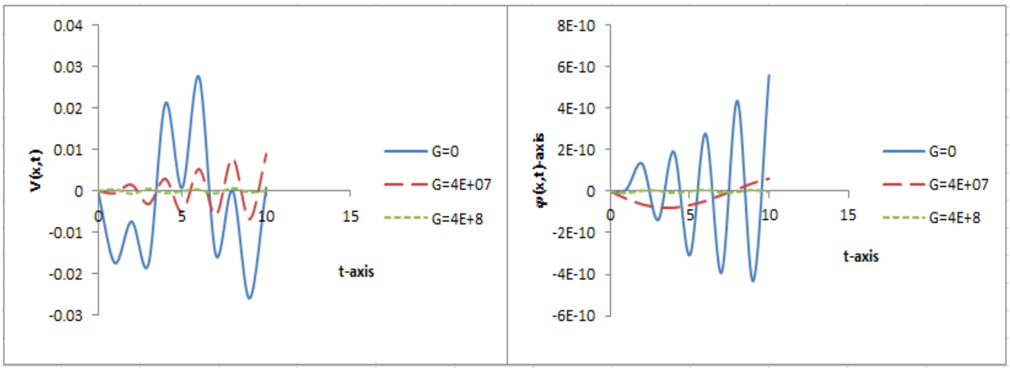


**Fig. 3. Transverse displacement & Rotation of the non-uniformly prestressed thick beam under distributed moving load travelling at varying velocity for various values of foundation modulus G and fixed values of N(40000) and K(40000) that traversed by moving distributed force.**

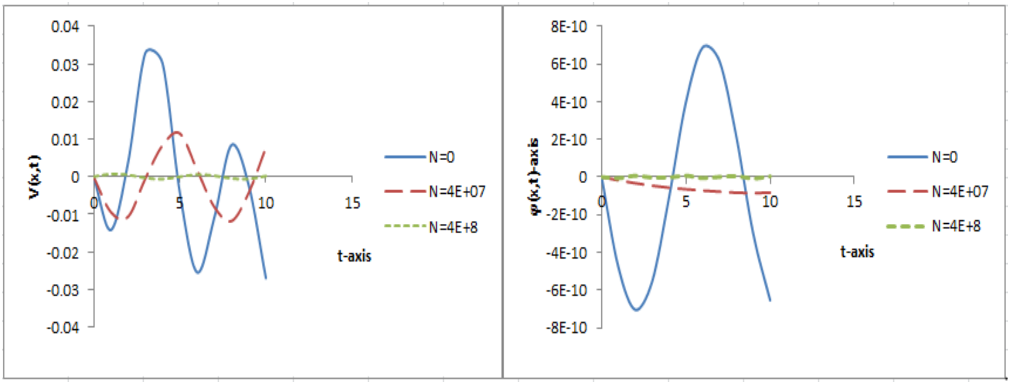


**Fig. 4. Transverse displacement & Rotation of the non-uniformly prestressed thick beam under distributed moving load travelling at varying velocity for various values of axial force N and for fixed values of G(77000) and K(40000) and traversed by moving distributed mass.**

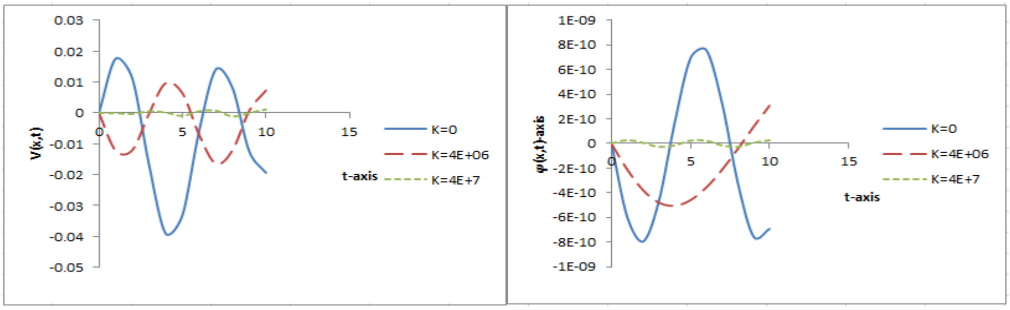


**Fig. 5. Transverse displacement & Rotation of the non-uniformly prestressed thick beam under distributed moving load travelling at varying velocity for various values of foundation stiffness K and fixed values of G(77000) and N(40000) that traversed by moving distributed mass.**

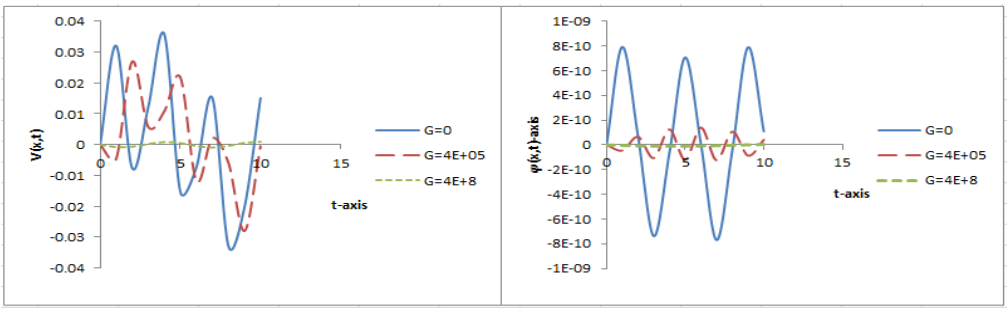


**Fig. 6. Transverse displacement & Rotation of the non-uniformly prestressed thick beam under distributed moving load travelling at varying velocity for various values of foundation modulus G and fixed values of N(40000) and K(40000) that traversed by moving distributed mass.**

Fig. 1, Fig. 2 and Fig. 3 shows the transverse displacement and rotation responses of a non-uniformly prestressed thick beam under distributed moving load travelling at varying velocity under the action of moving distributed force for various values of (i) axial force N and for fixed values of other parameters; (ii) various values of foundation stiffness K and for fixed values of other parameters and (iii) values of shear modulus G and for fixed values of other parameters. The result shows that as N, K and G increases, the deflection/rotation of the beam decreases. Similar results are obtained when the beam is subjected to moving mass as shown in fig.4, fig.5 and fig.6.

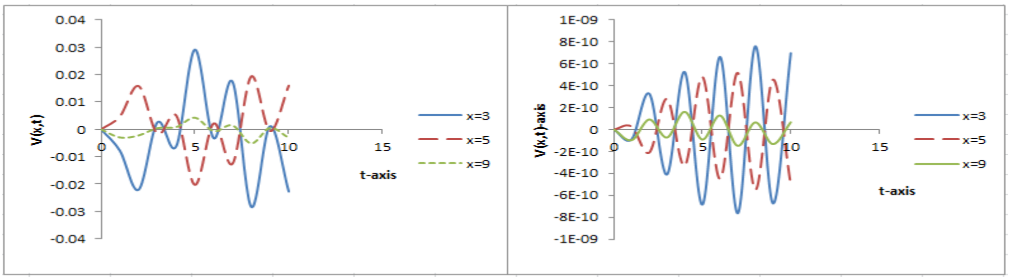


**Fig. 7. Transverse displacement & Rotation of the non-uniformly prestressed thick beam under distributed moving load travelling at varying velocity for various values of X and fixed values of N(40000), K(40000) and G(77000) that traversed by moving distributed force.**

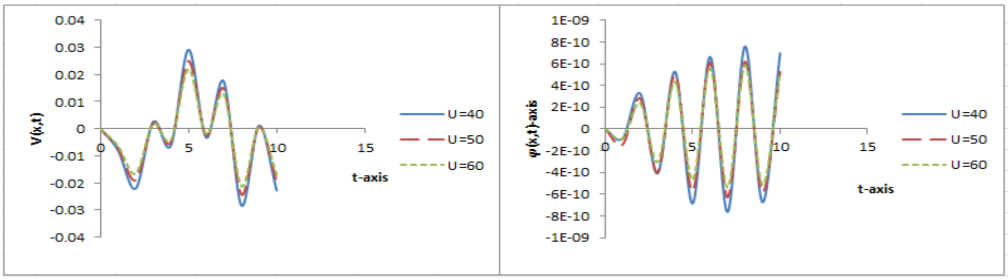


**Fig. 8. Transverse displacement & Rotation of the non-uniformly prestressed thick beam under distributed moving load travelling at varying velocity for various values of U and fixed values of N(40000), K(40000) and G(77000) that traversed by moving distributed force.**

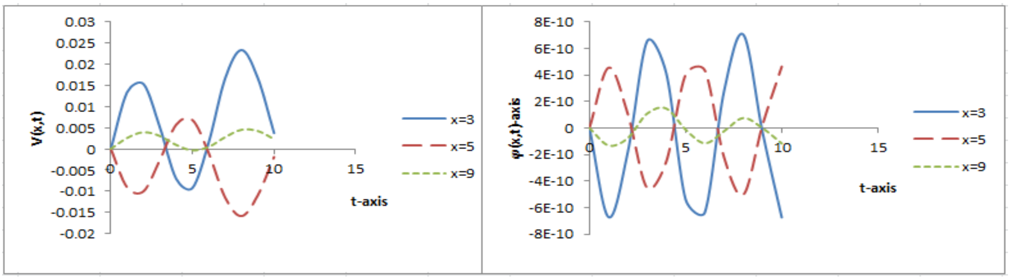


**Fig. 9. Transverse displacement & Rotation of the non-uniformly prestressed thick beam under distributed moving load travelling at varying velocity for various values of X and fixed values of N(40000), K(40000) and G(77000) that traversed by moving distributed mass.**

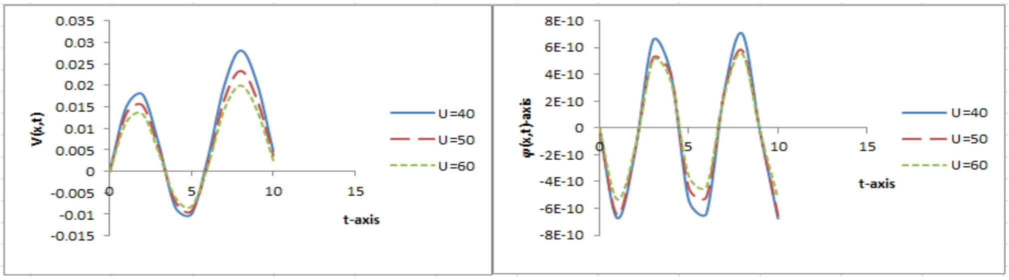


**Fig. 10. Transverse displacement & Rotation of the non-uniformly prestressed thick beam under distributed moving load travelling at varying velocity for various values of U and fixed values of N(40000), K(40000) and G(77000) that traversed by moving distributed mass.**

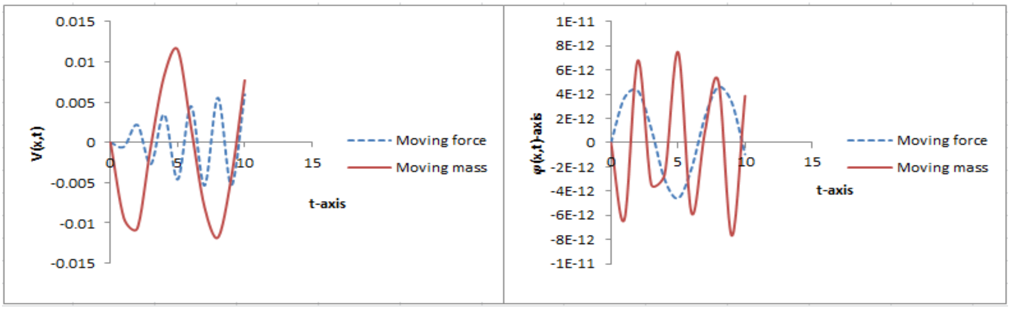


**Fig. 11. Comparison of the moving force and moving mass cases of Transverse displacement & Rotation of the non-uniformly prestressed thick beam under distributed moving load travelling at varying velocity for fixed values of N(40000), K(40000) and G(77000).**

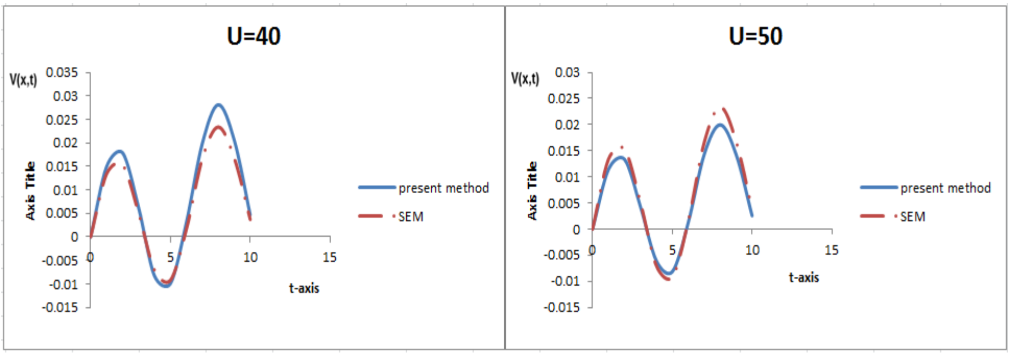


**Fig. 12. Comparison of the present method and SEM method U=40 & U=50 of the non-uniformly prestressed thick beam under distributed moving load travelling at varying velocity for fixed values of N(40000), K(40000) and G(77000).**

Fig. 7 shows the transverse displacement and rotation responses of a non-uniformly prestressed thick beam under distributed moving load travelling at varying velocity for various values of load position $x_o$ and for fixed values of N(40000), G(77000) and K(40000) and traversed by moving distributed force. The graph shows that as $x_o$ increases, the deflection and rotation of the beam decreases. Similar results are obtained when the simply supported thick beam is subjected to partially distributed mass travelling at varying velocity as shown in fig. 9. The displacement and rotation of the beam for various values of average velocity U at any point and for fixed values of axial force N, foundation stiffness K and shear modulus G for various travelling time t are shown in fig. 8. It is observed that as the value of K increases, the deflection/rotation of the beam decreases. Similar results are obtained when the beam is subjected to moving mass as shown in fig. 10. Finally, fig. 11 depicts the comparison of the transverse displacement of moving distributed force and the moving distributed mass cases. clearly, the response amplitude of moving distributed mass is greater than that of the moving distributed force problem. This important result has been reported in [12, 14, 15, 16, 17, 18], hence the inertia effects of a moving load must be considered when heavy loads are involved. It also shows that moving distributed force solution is not always an upper bound to the solution of a moving distributed mass problem.

# 6 Conclusion

In this paper, a procedure involving the Galerkin's method and integral transform technique has been used to solve the problem of a non-uniform beam when it is subjected to constant and harmonic variable magnitude moving loads. The objective is to study the behavior of the dynamical system. In particular, analytical solution in series form is obtained for the deflection of the elastic beam and the effects of foundation stiffness K and the axial force N on the vibrating system are investigated. Analytical solution and numerical result in plotted curves show that as the value of foundation stiffness K and axial force N increase, the deflection profile of the non-uniform beam decreases. Thus, in general, higher values of foundation stiffness K and axial force N reduce the risk of resonance in a dynamical system involving non-uniform beam under the action of a moving load. To verify the accuracy of the present method, the dynamic responses of a simply supported Timoshenko beam obtained by the present method and the frequency-domain spectral element method (SEM) are compared in figure 12 at two different velocities. The SEM is known as an exact element method that provides extremely accurate solutions to one dimensional structural dynamics problems [19]. Song et al. [20] applied the SEM to a moving problem to verify its high accuracy. Figure 12 shows that the dynamic responses obtained by the using the present method are almost identical to those obtained by using the SEM.

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# Competing Interests

Authors have declared that no competing interests exist.

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